21

Reprint from
JOURNAL OF APPLIED MATHEMATICS AND PHYSICS (ZAMP)
Vol. 17, Fasc. 2 (1966) BIRKHÄUSER VERLAG BASEL Pages 348–353

Effect of Viscosity upon Liquid Velocity in Axi-Symmetric Sheets

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Effect of Viscosity upon Liquid Velocity in Axi-Symmetric Sheets[1])

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Introduction

The motion of axi-symmetric liquid sheets has received continued attention over the past century-and-a-half. Much of this attention has been motivated by the relevance of this configuration to certain spray forming devices. TAYLOR [1][2]) summarizes and extends earlier work on the dynamics of sheets formed by the collision of free jets.

The previous analyses have generally regarded the effect of internal viscous shear to be negligible, although the contribution of external air drag has been treated in some detail [1]. In 1951 PORITSKY [2] discussed the shear stresses that arise in spherically symmetric radial flows as a result of tangential stretching. This study will show the effect of similar stresses upon the velocity of radially flowing liquid sheets.

[1]) This work has been supported in part by the College of Agriculture Research Center.

[2]) Numbers in brackets refer to References, page 352.

Formulation and Linearized Solution

Let an incompressible liquid of viscosity, ν, be supplied in an outward radial direction at velocity, u_0, on a boundary at radius, r_0. This would be the approximate result of the collision of two axial jets of radius, r_0, and velocity, u_0 (see Figure 1).

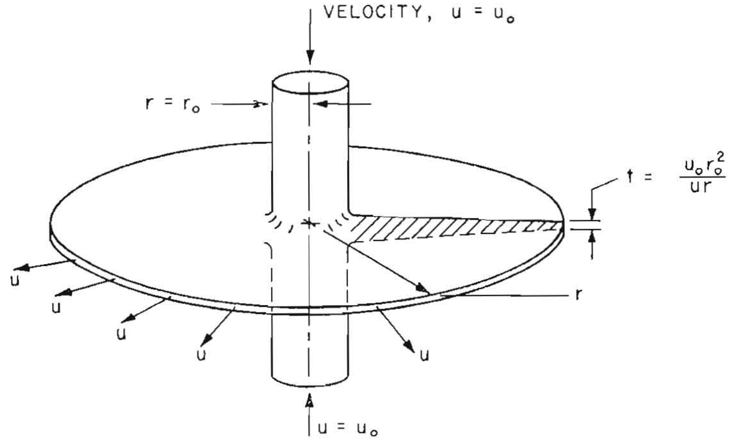

Figure 1
Configuration of axi-symmetric liquid sheet (special case in which liquid is supplied by colliding jets).

The continuity equation requires that the thickness, t, of the sheet be proportional to $(u\,r)^{-1}$ in general and equal to $u_0\,r_0^2/u\,r$ for the colliding jet configuration. The Navier-Stokes momentum equation:

$$u\,\frac{du}{dr} = \nu\left[\frac{d^2u}{dr^2} + \frac{1}{r}\,\frac{du}{dr} - \frac{u}{r^2}\right] \tag{1}$$

then specifies $u = u(r)$ independently of the continuity equation. We fix the following boundary conditions on Equation (1)

$$u(r_0) = u_0 \text{ and } 0 \leqslant u < \infty \text{ when } r \geqslant r_0\,. \tag{2}$$

Under the transformations:

$$\lambda = \frac{r_0\,u}{\nu} \text{ and } \eta = \frac{r}{r_0}$$

Equations (1) and (2) become:

$$\lambda'' + \frac{\lambda'}{\eta} - \frac{\lambda}{\eta^2} - \lambda\,\lambda' = 0 \tag{1a}$$

with boundary conditions:

$$\lambda(1) = Re \text{ and } 0 \leqslant \lambda < \infty \text{ when } \eta \geqslant 1 \tag{2a}$$

where $Re \equiv u_0\,r_0/\nu$, the Reynolds number of the flow.

A reasonable approximate solution for Equation (1a) can be obtained if the nonlinear term, $\lambda\,\lambda'$, is replaced with $Re\,\lambda'$. This should be valid if the velocity is not affected too strongly by viscosity. Accordingly Equation (1a) becomes:

$$\lambda'' + \frac{1 - Re\,\eta}{\eta}\,\lambda' + \frac{-1}{\eta^2}\,\lambda = 0\,. \tag{3}$$

This is exactly Frobenius' equation. Its general solution is:

$$\lambda = A\,\frac{1 + Re\,\eta}{\eta} + B\,\frac{\exp\,(Re\,\eta)}{\eta}\,. \tag{4}$$

The boundary conditions (2a) fix the constants, A and B, as $Re/(1 - Re)$ and zero, respectively. Thus the linearized result, λ_l, is:

$$\lambda_l = \frac{Re}{1+Re}\left(\frac{1+Re\,\eta}{\eta}\right). \tag{5}$$

Exact Solution of the Equation of Motion

Under the transformation

$$\lambda = -2\,s + \frac{1}{\eta} \tag{6}$$

Equation (1a) becomes:

$$s'' + 2\,s'\,s = \frac{1}{2\,\eta^3} \tag{7}$$

which can be integrated once to give the Riccatti equation:

$$s' + s^2 = -\left(\frac{1}{4\,\eta^2}\right) + C. \tag{8}$$

The constant, C, can be evaluated at the boundary, with the help of Equation (6):

$$C = \frac{Re^2 - 2\,Re - 2\,\lambda'(1)}{4}. \tag{9}$$

Another transformation, namely:

$$s = \frac{y'}{y} \tag{10}$$

transforms Equation (8) into:

$$y'' + \left(\frac{1}{4\,\eta^2} - \frac{C}{4}\right) y = 0. \tag{11}$$

The constant, C, depends upon the unknown initial slope, $\lambda'(1)$. First let us suppose that $\lambda'(1)$ is such that $C = 0$. Equation (11) then becomes EULER's equation, whose general solution,

$$y = (A + B\ln\eta)\,\eta^{1/2} \tag{12}$$

can be substituted into Equations (10) and (6) to give:

$$\lambda = \frac{-2\,B}{\eta\,[A + B\ln\eta]}. \tag{13}$$

Using the first boundary condition (2a), we find that $(B/A) = -Re/2$. This solution has a singularity at $\eta = \exp(2/Re)$, and will therefore violate the second boundary condition.

If, on the other hand, $C \neq 0$ then Equation (11) is a special form of BESSEL's equation. Its general solution is:

$$y = \sqrt{\eta}\,[A\,I_0(\sqrt{C}\,\eta) + B\,K_0(\sqrt{C}\,\eta)]. \tag{14}$$

Substitution of this result into Equations (10) and (6) gives the general solution of Equation (1a):

$$\lambda = -2\sqrt{C}\left[\frac{A\,I_1(\sqrt{C}\,\eta) - B\,K_1(\sqrt{C}\,\eta)}{A\,I_0(\sqrt{C}\,\eta) + B\,K_0(\sqrt{C}\,\eta)}\right]. \tag{15}$$

The functions I_0, I_1, K_0, and K_1 are all positive and monotonic. Since I_0 and I_1 increase without bound with their arguments, while K_0 and K_1 decrease to zero, we conclude that A must be zero to accommodate the second boundary condition. The applicable solution is thus:

$$\lambda = 2\sqrt{C}\,\frac{K_1(\sqrt{C}\,\eta)}{K_0(\sqrt{C}\,\eta)}. \tag{16}$$

Substitution of the first boundary condition gives the following transcendental equation for C:

$$Re = 2\sqrt{C}\,\frac{K_1(\sqrt{C})}{K_0(\sqrt{C})}\,. \tag{17}$$

Equations (16) and (17) provide the particular solution to our problem.

The asymptotic behavior of K_1 and K_0 is such that λ, as given by Equation (16), very rapidly approaches its limiting value, λ_∞, as η increases:

$$\lambda_\infty = 2\sqrt{C}. \tag{18}$$

Comparison of Exact and Linearized Results

Figure 2 displays typical integral curves for Equation (1a), satisfying the first boundary condition for $Re = 2$. The exact solution of the present problem, which we have shown to be uniquely determined in the form of Equation (16) by the boundary conditions, is included as a member of this family of solutions. The linearized solution is also displayed in this figure. It compares very well with the exact result, even at this low Reynolds number.

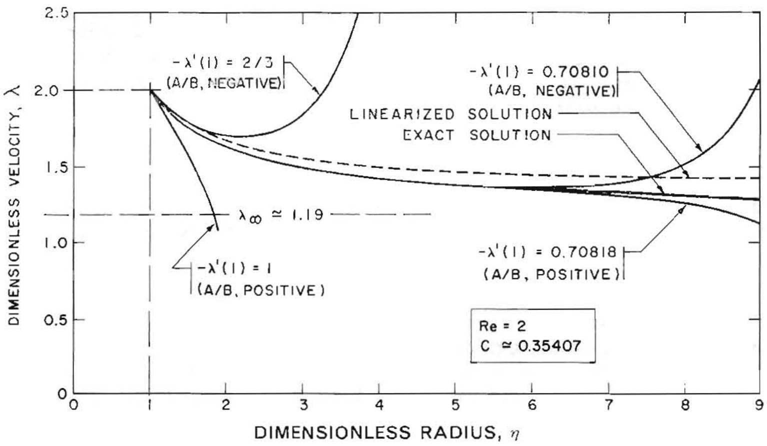

Figure 2

Comparison of solutions of the momentum equation.

As a matter of interest, four incorrect values of initial slope are shown. Two of these result from the use of positive values of A/B in Equation (15), and give solutions that drop toward negative asymptotic values. Two result from negative values of A/B and give solutions that approach positive infinity as η approaches the zero of the denominator of Equation (15).

Figure 3 compares the exact values of λ_∞/Re given by Equations (18) and (17) with those given by the linearized equation:

$$\lambda_{l\infty} = \frac{Re^2}{Re + 1}\,. \tag{19}$$

Except for very low Reynolds numbers, this simple approximation is quite accurate. At large Reynolds numbers, Equations (19) and (18) both give:

$$\frac{Re - \lambda_\infty}{Re} = \frac{u_0 - u_\infty}{u_0} = \frac{1}{Re}\,. \tag{20}$$

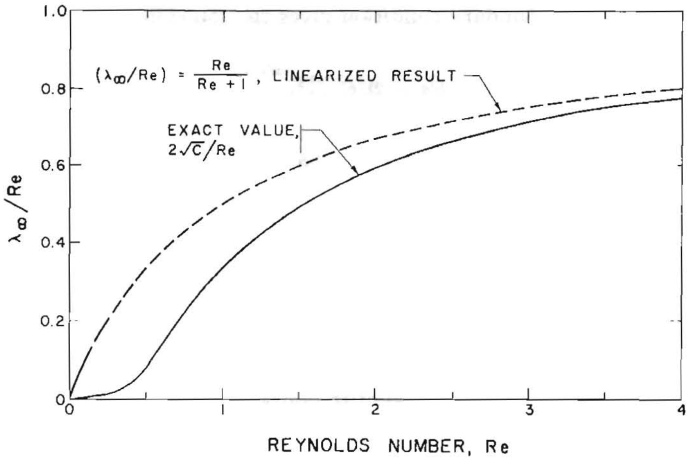

Figure 3

Comparison of exact and linearized evaluations of λ_∞.

The Reynolds number is seldom small in real physical systems. The Reynolds number of a 1-mm orifice discharging water at 6 m/sec would, for example, be about 3000. The substitution of glycerin would reduce this figure to about 4.5. In either case the linearization is quite accurate. So too is Equation (20).

Acknowledgments

The authors are indebted to Prof. S. C. Lowell of the Department of Mathematics for his substantial counsel and to the Computing Center for processing computations.

NOMENCLATURE

A, B arbitrary constants,
C constant of integration, evaluated in Equation (9),
I_0, I_1 modified Bessel functions of the first kind, of zeroth and first order, respectively,
K_0, K_1 modified Bessel functions of the second kind, of zeroth and first order, respectively,
Re Reynolds number of the flow, $r_0\, u_0/\nu$,
r radial coordinate,
s $(\eta^{-1} - \lambda)/2$,
t thickness of sheet,
u velocity of sheet,
y transformed s, given by Equation (10),
η dimensionless radial coordinate, r/r_0,
λ dimensionless velocity, $u\, r_0/\nu$,
ν kinematic viscosity.

General Subscripts

l denoting a linearized result,
o denoting evaluation made at inner boundary of sheet,
∞ denoting an evaluation made as $\eta \to \infty$.

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[1] G. I. Taylor, *The Dynamics of Thin Sheets of Fluid*, Proc. Roy. Soc., [A] *253*, 289 (1959).
[2] H. Poritsky, *The Collapse or Growth of a Spherical Bubble or Cavity in a Viscous Fluid*, 1st U.S. Congress of Appl. Mech., 1951, p. 813.