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Randomly Oscillating Turbulent Channel Flows[1])

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Introduction

A knowledge of the relationships between pressure and velocity pulsations in conduits is important in many current problems of fluid metering and control. TIEN and LIENHARD [1][2]) first presented relationships between the characteristics of a spectrum of pressure oscillations, and the spectrum of resulting velocity oscillations, in laminar pipe and channel flows. This note will extend that work to describe the far more frequently encountered case of turbulent flow.

The previous paper showed that solution of the Navier-Stokes equations gave velocity responses of the form:

$$u(y, t) = \sum_{n=0}^{\infty} \frac{K_n}{n} (A \sin n\, t + B \cos n\, t)\,, \tag{1}$$

to pressure spectra of the complex form:

$$p_x \equiv -\frac{1}{\varrho} \frac{dp}{dx} = \sum_{n=0}^{\infty} K_n\, e^{i n t}\,, \tag{2}$$

where A and B were cumbersome functions of the transverse coordinate, y, of the conduit, and of the frequency of oscillation, n.

The characteristics of the pressure and velocity oscillations were then related to one another with the statistical relations [2]:

$$\overline{u^2(y)} = \overline{p_x^2} \int_0^{\infty} \left[\frac{A^2(y, n) + B^2(y, n)}{n^2}\right] f(y, n)\, dn\,, \tag{3}$$

1) This study was supported in part by the R. L. Albrook Hydraulic Laboratory of the Wash. State Univ. Division of Industrial Research.

2) Numbers in brackets refer to References, page [redacted].

THE MISSING SYMBOL SHOULD BE A LOWER-CASE XI

$R(y, \xi)$

$$f(y, n) = \frac{1}{\pi} \int_{-\infty}^{\infty} R_p(y, \xi) \cos(n\,\xi)\, d\xi\,, \tag{4}$$

$$R_u(y, \text{[redacted]}) = \frac{\overline{p_x^2}}{u^2(y)} \int_0^{\infty} \left[\frac{A^2(y, n) + B^2(y, n)}{n^2}\right] f(y, n) \cos(n\,\xi)\, dn\,, \tag{5}$$

$$g(y, n) = \frac{\overline{p_x^2}}{u^2(y)} \left[\frac{A^2(y, n) + B^2(y, n)}{n^2}\right] f(y, n)\,. \tag{6}$$

The resulting dimensionless 'spectral transfer function':

$$\Phi \equiv \frac{A^2 + B^2}{n^2} \frac{\nu^2}{a^4} = \frac{A^2 + B^2}{\eta^2}\,, \tag{7}$$

was prohibitively complicated for use in Equations (3), (4), (5) and (6). However, a very simple and accurate approximation to Φ, namely:

$$\Phi \simeq \frac{1 - \exp(-I\,\eta^2)}{\eta^2}\,, \tag{8}$$

where:

$$I \equiv \operatorname*{Limit}_{\eta \to 0} \Phi\,, \tag{9}$$

facilitated the application of these relations.

Analysis of the Turbulent Flow Case

The dynamic equation for the turbulent flow case can be written in the form:

$$\frac{\partial u}{\partial t} = p_x + \frac{\partial}{\partial y}\left[(\nu + \varepsilon)\frac{\partial u}{\partial y}\right], \tag{10}$$

with boundary conditions:

$$u(0, t) = \left.\frac{\partial u}{\partial y}\right|_{y=a} = 0\,. \tag{11}$$

The eddy diffusivity, ε, in Equation (10) is generally a complicated function of position and of velocity. It is also an implicit function of time in the present case. The complex character of ε will render Equation (10) both nonlinear and intractable, and the following approximations must be made to obtain a solution:

Since relatively small oscillations are usually superposed on a relatively large steady flow component, it is reasonable to base ε upon the steady component and to neglect the contribution of the oscillations. This will make ε independent of time.

In accordance with the conventional semi-empirical theory [3], ε or $(\nu + \varepsilon)$ will be regarded as a function of position and of the gross velocity field (instead of the local velocity field). Thus, for any Reynolds number based upon the average flow velocity, $(\nu + \varepsilon)$ will only depend upon y.

Experiments reveal that for turbulent flows that are steady in the gross sense, $(\nu + \varepsilon)$ should vary in a roughly parabolic fashion with y-possibly exhibiting a minor dip at the centerline [3]. It is thus reasonable to propose, as a first approximation, the linear relationship:

$$\nu + \varepsilon = \nu + b\,y\,. \tag{12}$$

It will be shown subsequently that this assumption of a linear relationship between ε and y is not necessarily of first importance, since other assumptions will lead to a common method for correlating experimental information.

The solution for Equation (10) will be assumed to be separable:

$$u = \sum_{n=0}^{\infty} e^{int} \lambda(y) \, . \tag{13}$$

Then Equations (10) and (11) become:

$$z \lambda''(z) + \lambda'(z) - \frac{i n}{b^2} = - \frac{K_n}{b^2} \, , \tag{14}$$

and:

$$\lambda' (v + b a) = \lambda(v) = 0 \, . \tag{15}$$

where $z = v + b y$. Equation (14) becomes Bessel's equation under the substitution, $w = 2\sqrt{n z}/b$, and the resulting expression for $u (y, t)$ is:

$$u(y, t) = \sum_{n=0}^{\infty} e^{int} \left(- \frac{i K_n}{n} \right) \left\{ 1 - \frac{J_0 (i^{3/2} w)}{J_0 (i^{3/2} \sqrt{\eta_t}) - Y_0 (i^{3/2} \sqrt{\eta_t}) \dfrac{J_1 (i^{3/2} \sqrt{\phi})}{Y_1 (i^{3/2} \sqrt{\phi})}} - \frac{Y_0 (i^{3/2} w)}{Y_0 (i^{3/2} \sqrt{\eta_t}) - J_0 (i^{3/2} \sqrt{\eta_t}) \dfrac{Y_1 (i^{3/2} \sqrt{\phi})}{J_1 (i^{3/2} \sqrt{\phi})}} \right\} \tag{16}$$

where:

CLOSE PARENTHESES

$$\eta_t \equiv \frac{4 n v}{b^2} \quad \text{and} \quad \phi \equiv \frac{4 n (v + b a)}{b^2} \, .$$

The real part of Equation (16) is:

$$u(y, t) = \sum_{n=0}^{\infty} \frac{K_n}{n} \left\{ \left(1 - \frac{L_1 L_3 + L_2 L_4}{L_3^2 + L_4^2} \right) \sin n t - \left(\frac{L_2 L_3 - L_1 L_4}{L_3^2 + L_4^2} \right) \cos n t \right\} , \tag{17}$$

where:

$$L_1 = ber\, w \, ker_1 \sqrt{\phi} + kei\, w \, bei_1 \sqrt{\phi} - bei\, w \, kei_1 \sqrt{\phi} - ker\, w \, ber_1 \sqrt{\phi} \, , \tag{18}$$

$$L_2 = bei\, w \, ker_1 \sqrt{\phi} + ber\, w \, kei_1 \sqrt{\phi} - kei\, w \, ber_1 \sqrt{\phi} - ker\, w \, bei_1 \sqrt{\phi} \, , \tag{19}$$

$$L_3 = ber \sqrt{\eta_t} \, ker_1 \sqrt{\phi} - bei \sqrt{\eta_t} \, kei_1 \sqrt{\phi} - ker \sqrt{\eta_t} \, ber_1 \sqrt{\phi} - kei \sqrt{\eta_t} \, bei_1 \sqrt{\phi} \, , \tag{20}$$

$$L_4 = bei \sqrt{\eta_t} \, ker_1 \sqrt{\phi} + ber \sqrt{\eta_t} \, kei_1 \sqrt{\phi} - kei \sqrt{\eta_t} \, ber_1 \sqrt{\phi} - ker \sqrt{\eta_t} \, bei_1 \sqrt{\phi} \, , \tag{21}$$

The centerline and average velocities, $u_{c.l.}$ and u_{avg}, are of particular interest. The former is given by Equation (17) evaluated at $w = \sqrt{\phi}$. The latter is defined by:

$$u_{avg} = \frac{1}{a} \int_0^a u \, dy \, , \tag{22}$$

which integration leads to:

$$u_{avg}(t) = \sum_{n=0}^{\infty} \frac{K_n}{n} \left\{ \left(1 - \frac{2}{(h^2-1)\,\eta_t} \, \frac{L_3 L_5 + L_4 L_6}{L_3^2 + L_4^2} \right) \sin n\,t \right.$$
$$\left. - \left(\frac{2}{(h^2-1)\,\eta_t} \, \frac{L_3 L_6 - L_4 L_5}{L_3^2 + L_4^2} \right) \cos n\,t \right\}. \tag{23}$$

The new functions L_5 and L_6 are:

$$\left.\begin{aligned} L_5 = {} & ker_1\sqrt{\phi}\left(\sqrt{\phi}\,bei'\sqrt{\phi} - \sqrt{\eta_t}\,bei'\sqrt{\eta_t}\right) - bei_1\sqrt{\phi}\left(\sqrt{\phi}\,ker'\sqrt{\phi} - \sqrt{\eta_t}\,ker'\sqrt{\eta_t}\right) \\ & + kei_1\sqrt{\phi}\left(\sqrt{\phi}\,ber'\sqrt{\phi} - \sqrt{\eta_t}\,ber'\sqrt{\eta_t}\right) - ber_1\sqrt{\phi}\left(\sqrt{\phi}\,kei'\sqrt{\phi} - \sqrt{\eta_t}\,kei'\sqrt{\eta_t}\right) \end{aligned}\right\} \tag{24}$$

DELETE EXTRA LINE

and:

$$\left.\begin{aligned} L_6 = {} & -ker_1\sqrt{\phi}\left(\sqrt{\phi}\,ber'\sqrt{\phi} - \sqrt{\eta_t}\,ber'\sqrt{\eta_t}\right) + kei_1\sqrt{\phi}\left(\sqrt{\phi}\,bei'\sqrt{\phi} - \sqrt{\eta_t}\,bei'\sqrt{\eta_t}\right) \\ & + ber_1\sqrt{\phi}\left(\sqrt{\phi}\,ker'\sqrt{\phi} - \sqrt{\eta_t}\,ker'\sqrt{\eta_t}\right) - bei_1\sqrt{\phi}\left(\sqrt{\phi}\,kei'\sqrt{\phi} - \sqrt{\eta_t}\,kei'\sqrt{\eta_t}\right), \end{aligned}\right\} \tag{25}$$

where:

$$h \equiv \sqrt{\frac{\phi}{\eta_t}} = \sqrt{1 + \frac{b\,a}{\nu}}. \tag{26}$$

As h varies between 1 and ∞, all possible linear increases of $(\nu + \varepsilon)$ from the wall to the centerline are represented. When $h \to \infty$ the present solution becomes indeterminate and should be abandoned in favor of the previous laminar flow solution.

Both $u_{c.l.}$ and u_{avg} are in the form of Equation (1); thus A and B can be identified and the spectral transfer function defined as:

$$\Phi \equiv \frac{A^2 + B^2}{n^2} \, \frac{b^4}{4\,d^2} = \frac{A^2 + B^2}{\eta_t^2}. \tag{27}$$

Figures 1 and 2 show $\Phi_{c.l.}$ and Φ_{avg} for h equal to 1.5, 2, and 4.

As in the laminar case, the extremely intractable function Φ can be closely approximated with:

$$\Phi \simeq \frac{1 - \exp(-I\,\eta_t^2)}{\eta_t^2}. \tag{7a}$$

This expression is exact in the limits as $\eta_t \to 0$ and ∞. In particular the left hand limit is:

$$I_{c.l.} = m^2\,h^4\,, \tag{28}$$

or:

$$I_{avg} = \left[\frac{d}{c}\left(\frac{d}{c} - 2\,m\right) + m^2\right] h^4\,, \tag{29}$$

where:

$$m \equiv \frac{1}{4}\left[\frac{h^2 - 1}{2} - \ln h\right], \tag{30}$$

$$c \equiv \frac{h^2 - 1}{2}, \tag{31}$$

$$d \equiv \frac{h^4 - 1}{16} - \frac{h^2 \ln h}{4}. \tag{32}$$

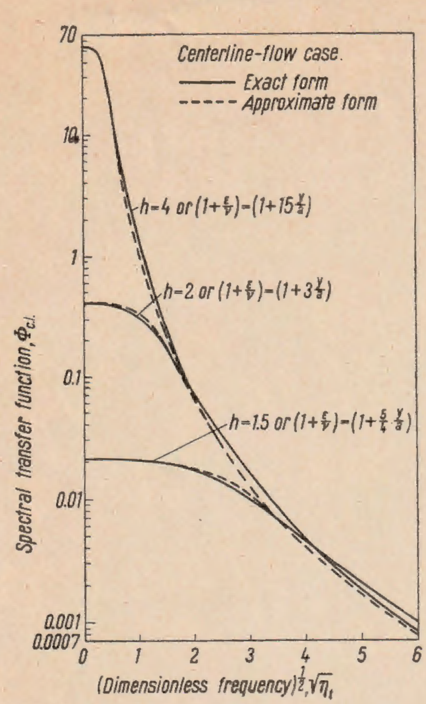

Figure 1
Exact and approximate forms of Φ
for centerline flow in a channel.

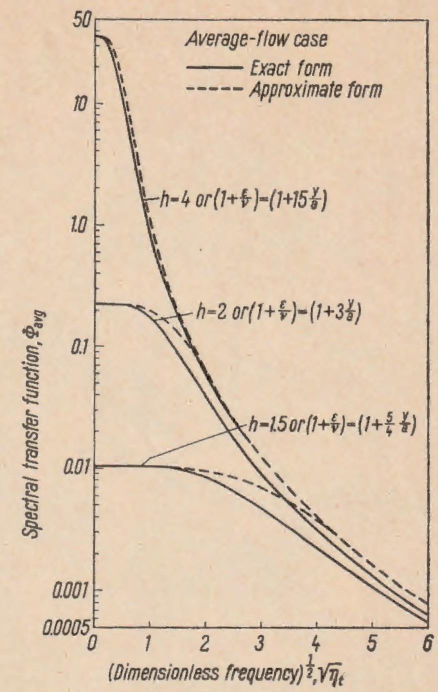

Figure 2
Exact and approximate forms of Φ
for average flow in a channel.

The approximation to Φ is included in Figures 1 and 2. The worst deviations of the approximation from the points have been computed are about 60% and 25% for Φ_{avg} and $\Phi_{c.l.}$, respectively. THAT

Equations (7) and (7a) probably provide an adequate description of the spectral transfer function for the pressure-flow characteristics of all oscillating parallel flow systems for which the physical properties are independent of u and p. Accordingly, one experiment made at low frequency will serve to evaluate I in a real system for which the variation of $(\nu + \varepsilon)$ with y, of the geometry, are more complicated than treated here. Equations (7) and (7a) can then be used to obtain pressure-flow information over the entire frequency range. The method of application of Equations (7) and (7a) is exactly the same as in the laminar case, and is illustrated in the paper [1] on laminar oscillating flow.

Nomenclature

- A, B coefficients defined by comparison of Equation (1) with the appropriate dynamical equation for u,
- a half width of channel, or radius of pipe,
- b rate of increase of ε with y,
- c defined in Equation (31),
- d defined in Equation (32),
- f normalized spectral density of pressure oscillations,
- g normalized spectral density of velocity oscillations,
- h turbulent viscosity parameter defined in Equation (26),

I $\underset{\eta \to 0}{\text{Limit}}\ \Phi$

K_n amplitude of the individual pressure oscillations,

$L_1, L_2, L_3, L_4, L_5, L_6$ functions defined by Equations (18) (19) (20) (21) (24) (25),

m defined in Equation (30),

n frequency of pressure and velocity oscillations,

p pressure,

p_x defined in Equation (2),

R_p autocorrelation coefficient for pressure oscillations,

R_u autocorrelation coefficient for velocity oscillations,

t time,

u axial velocity,

w variable defined after Equation (15),

x axial position coordinate,

y transverse position coordinate,

z variable defined after Equation (15),

ε eddy diffusivity for momentum,

η dimensionless frequency, defined in Equation (7), for laminar case,

η_t dimensionless frequency, defined after Equation (16), for turbulent case,

λ function defined in Equation (13),

ν laminar kinematic viscosity,

ξ variable displacement in time,

ϱ fluid density,

Φ spectral transfer function,

ϕ variable defined after Equation (16).

REFERENCES

[1] C. L. Tien and J. H. Lienhard, *Pressure-Flow Characteristics of Randomly Oscillating Pipe Flows*, Jour. App. Mech. *28*, No. 3, 463 (Sept. 1961).

[2] See, for example, S. H. Crandall (editor), *Random Vibration* (Technology Press of M.I.T., 1958), pp. 77–90.

[3] See, for example, J. G. Knudsen and D. L. Katz, *Fluid Dynamics and Heat Transfer* (McGraw-Hill Book Co., Inc. 1958), pp. 437–440.

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Abstract

The velocity response to a sinusoidally oscillating pressure gradient is obtained for turbulent flow in a two-dimensional channel. The statistical relations between a spectrum of such pressure pulsations and the resulting spectrum of velocity oscillations are then written in terms of a spectral transfer function. This function, which incorporates the dynamic solution is complicated, but is well approximated with a simple expression.

Zusammenfassung

Für turbulente Strömung in einem zweidimensionalen Kanal wird die Geschwindigkeitsverteilung, die durch einen sinusoidal oszillierenden Druckgradient verursacht ist, erhalten. Die statistischen Beziehungen zwischen einem Spektrum von solchen Druckpulsationen und das resultierende Spektrum der Geschwindigkeitsoszillationen werden dann in Abhängigkeit von einer spektralen Übertragungs-Funktion geschrieben. Diese Funktion, die die dynamische Lösung einbezieht, ist kompliziert; sie lässt sich aber durch einen einfachen Ausdruck gut annähern.

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