Effects of Gravity and Surface Tension Upon Liquid Jets Leaving Poiseuille Tubes

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A method is developed for approximating the transverse velocity component in the incompressible two-dimensional boundary layer equations. The method is restricted to flows that are symmetrical in the transverse coordinate, and it facilitates easy numerical integration in such problems as the prediction of jet and wake flows. The prediction of velocity profiles and diameters for free exiting Poiseuille flows, under the influence of both gravity and surface tension, is then undertaken. Analytical results obtained by the present method are found to agree very closely with experiments. The experiments also show that, in low Weber number situations, contact angle at the exit plane can dominate the early relaxation of the exit profile.

Part I—Approximate Method of Solution of Equations

Introduction

We shall first describe an approximate method for solving the boundary-layer equations and then use it to determine the behavior of exiting Poiseuille flows under a variety of conditions.

The solution is accomplished by eliminating the transverse velocity from the incompressible two-dimensional boundary-layer equations, with the aid of the continuity equation. This leaves a single, second-order equation in a single dependent variable. The approximation of the transverse velocity component is similar to Oseen's approximation in slow flows; but the inaccuracy of the resulting solution will be less than the original approximation to the transverse velocity component.

The value of the method lies in that it makes certain difficult problems solvable by a simple numerical integration. Since the method includes a limitation on boundary conditions that generally restricts it to symmetry in the transverse coordinate, it can be used to describe such configurations as jets and wakes.

This approximation is a generalization of a device used by Brun and Lienhard [1][1] in a recent study of the behavior of free jets leaving Poiseuille tubes. The generalization will make it possible to include effects of gravity and surface tension in this problem, and hopefully to treat other similar problems.

[1] Numbers in brackets designate References at end of paper.

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Development

We shall nondimensionalize the variables as follows:

$$u = u_p/U_0; \quad v = v_p \mathrm{Re}/U_0; \quad P = P_p/\rho U_0^2$$
$$x = x_p/L\mathrm{Re}; \quad y = y_p/L \quad \text{or} \quad r = r_p/L \tag{1}$$

The subscript p denotes the physical variables, and the dimensionless variables are unsubscripted. Unexplained terms are described in the Nomenclature section.

The two-dimensional boundary-layer equations with pressure gradient and gravity forces become, under substitution of equations (1)

$$uu_x + vu_y = u_{yy} - \left(\frac{dP}{dx} - \mathrm{Re/Fr}\right) \tag{2}$$

$$u_x = -v_y \tag{3}$$

Likewise, we obtain for the axisymmetric case

$$uu_x + vu_{\cdot} = [(ru_r)_r]/r - \left(\frac{dP}{dx} - \mathrm{Re/Fr}\right) \tag{4}$$

$$u_x = -[rv]_r/r \tag{5}$$

We now wish to devise a reasonable approximation for v in the

Nomenclature

$f(x)$ = arbitrary function of x
Fr = Froude number, U_0^2/gL
g = acceleration of gravity
L = characteristic length
P = dimensionless pressure, $P_p/\rho U_0^2$
r = dimensionless radial coordinate, $r_p/L\mathrm{Re}$
R = radius of a tube; R is L for exiting Poiseuille flow
R_c = radius of curvature in an axial plane (see Fig. 1)
Re = Reynolds number, U_0L/ν
$\bar{U}_e$ = average velocity of liquid leaving a tube; equal to U_0 for exiting Poiseuille flow
U_0 = characteristic velocity
U_∞ = free-stream velocity
u = dimensionless axial velocity, u_p/U_0
$\bar{u}$ = dimensionless average axial velocity
v = dimensionless transverse or radial velocity, $v_p \mathrm{Re}/U_0$
We = Weber number, $2R\bar{U}_e^2\rho/\sigma$
x = dimensionless axial position, $x_p/L\mathrm{Re}$
y = dimensionless transverse (or radial) position, y_p/L
Δ = dimensionless outer radius of a free jet
δ = boundary-layer thickness at trailing edge of a flat plate
ρ = density
σ = surface tension
ν = kinematic viscosity
p = general subscript denoting the physical, or dimensional, counterpart of a dimensionless variable

momentum equation. We shall do this by first solving it (with the continuity equation) for v, using an Oseen type of linearization [2] for the inertia term

$$uu_x + vu_y \simeq (1)\, u_x + (0)u_y$$

This amounts to replacing u_p with the characteristic velocity, U_0, and v_p with the center-line velocity (zero) since we are interested in symmetrical flows. The resulting momentum equations are then

$$u_x = u_{yy} - \left(\frac{dP}{dx} - \mathrm{Re/Fr}\right) \qquad (2a)$$

$$u_x = [(ru_r)_r]/r - \left(\frac{dP}{dx} - \mathrm{Re/Fr}\right) \qquad (4a)$$

Equations (2a) and (3) can be combined,

$$u_{yy} = -v_y + \left(\frac{dP}{dx} - \mathrm{Re/Fr}\right) \qquad (6)$$

and the result can be integrated with respect to y to get

$$u_y = -v + f(x) + y\left(\frac{dP}{dx} - \mathrm{Re/Fr}\right) \qquad (7)$$

Equations (4a) and (5) can likewise be combined and the results integrated.

$$u_r = -v + f(x)/r + \frac{r}{2}\left(\frac{dP}{dx} - \mathrm{Re/Fr}\right) \qquad (8)$$

The restriction of present considerations to flows that are symmetrical in y or r means that u_y or u_r, and v, must vanish on the centerline. The undetermined function $f(x)$ is therefore equal to zero and the approximations to v become

$$v = -u_y + y\left(\frac{dP}{dx} - \mathrm{Re/Fr}\right) \quad \text{or}$$

$$v = -u_r + \frac{r}{2}\left(\frac{dP}{dx} - \mathrm{Re/Fr}\right) \qquad (9)$$

Equations (9) give the approximate forms of v that we shall use to simplify the momentum equation. They will be in error in a way that we can assess qualitatively by looking at the effect of the simplified convective acceleration in equations (2a) or (4a). This error will only enter in the coefficient of one term in equations (2) or (4) after we have eliminated v using equations (9). Equation (2) then becomes

$$uu_x - u_y^2 = u_{yy} - \left(\frac{dP}{dx} - \mathrm{Re/Fr}\right)(1 + yu_y) \qquad (10)$$

and equation (4) becomes

$$uu_x - u_r^2 = [(ru_r)_r]/r - \left(\frac{dP}{dx} - \mathrm{Re/Fr}\right)(1 + ru_r/2) \qquad (11)$$

If equations (9) give even moderately reliable representations of v, the solution of equations (10) and (11) should be a fairly close approximation to u.

The full approximation of the convective acceleration used in equations (2a) and (4a) has been used in the past in this kind of problem. Niels Bohr [3] used it directly to get a coarse estimate of the rate of decay of exiting Poiseuille flows. Schetz [4] used it recently as the basis for an entirely different kind of approximate solution which he applied to boundary layers.

Equations (10) and (11), which are of second order in the single dependent variable, u, can easily be solved numerically for u^2 when they are cast in the following form:

$$(u^2)_x = 2\left[u_y^2 + u_{yy} - \left(\frac{dP}{dx} - \mathrm{Re/Fr}\right)(1 + yu_y)\right] \qquad (10a)$$

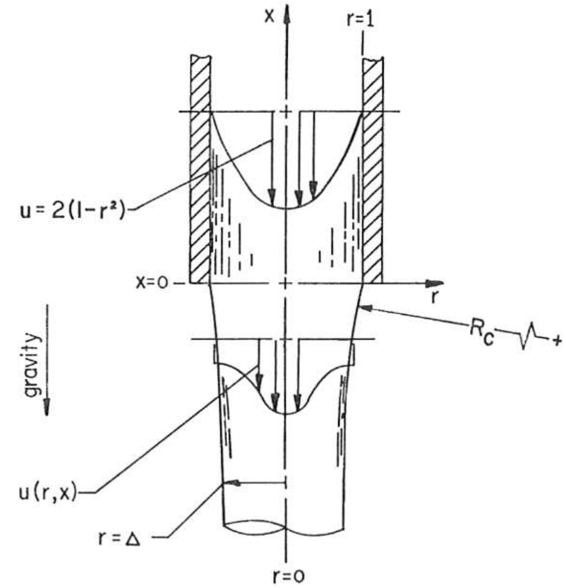


Fig. 1 Configuration of a free jet leaving a Poiseuille tube

$$(u^2)_x = 2\left[u_r^2 + [(ru_r)_r]/r - \left(\frac{dP}{dx} - \mathrm{Re/Fr}\right)(1 + ru_r/2)\right] \qquad (11a)$$

The alternative of formulating and solving the complete nonlinear partial differential equations for such problems, analytically or numerically, can be prohibitively complicated. The problem of an exiting Poiseuille jet flow becomes doubly complicated since the boundary cannot be specified until after the flow within it is known. The present approximation will accordingly be valuable if it is sufficiently accurate.

Exiting Poiseuille Flow

Brun and Lienhard treated the free jet leaving a Poiseuille tube as shown in Fig. 1, although they ignored the gravity term shown in the figure. They discovered that the analytical solution of equation (4a) for this case happened to have the property that $u_r = -v$. Accordingly they used a form of equation (11) in which $\left(\frac{dP}{dx} - \mathrm{Re/Fr}\right)$ was zero and the boundary conditions were:

$$u_r = 0 \quad \text{at } r = 0 \text{ and } \Delta; \quad u = 2(1 - r^2) \quad \text{at } x = 0 \qquad (12)$$

The outer boundary, $r = \Delta$, was found by stretching r and u to satisfy the condition that the mass and momentum flows at each downstream station had to be the same as at the exit plane.

Mass:

$$\int_0^\Delta ru\,dr = 1/2 \qquad (13)$$

Momentum:

$$\int_0^\Delta ru^2dr = 2/3 \qquad (14)$$

Conditions (13) and (14) effectively made an *integral method* of the approximation since the velocity profiles given by equation (11a) were stretched to accommodate them.

Before considering this solution, we might anticipate how it will behave. For this flow, uu_x will be negative and less than u_x near the center; it will be positive and less than u_x as r goes toward Δ. The term vu_y will be negative everywhere and it will vanish at the boundaries. The first approximation will therefore overestimate the inertia term everywhere, and profile will change less rapidly in the approximation than in the actual flow. The second approximation will then be in error to the extent of a

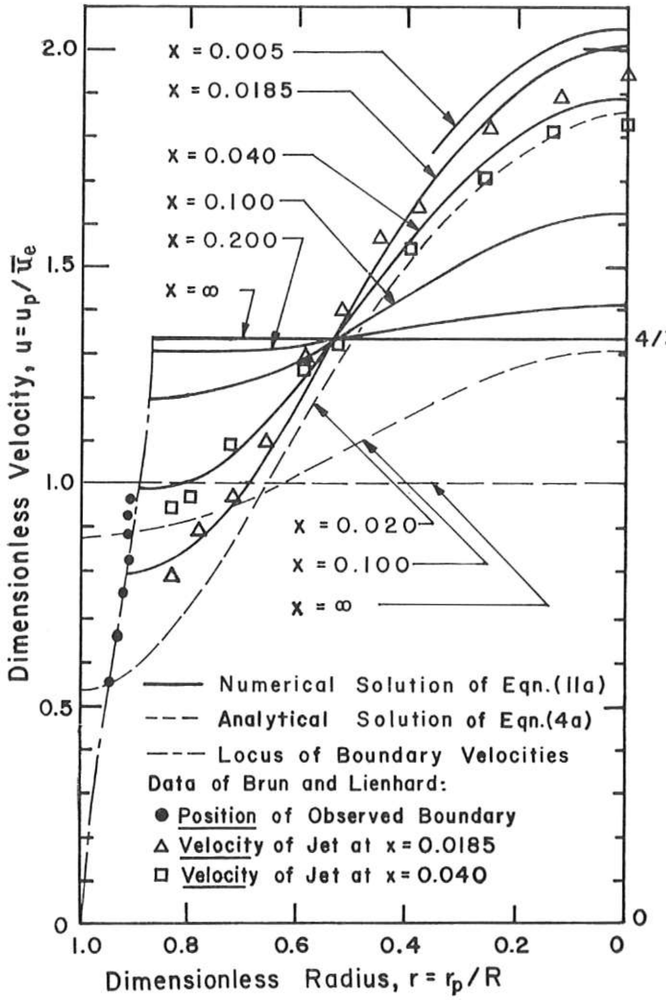


Fig. 2 Comparison of first and second levels of approximation for exiting Poiseuille flow

slight underestimation of v in the second inertia term. We would normally expect this to slow the relaxation of the flow; however, the imposition of integral conditions (13) and (14) will serve to compensate this error.

Fig. 2 compares Brun's and Lienhard's solutions with their data; it shows that the prediction is quite accurate. The figure also shows the relatively coarse solution of equation (4a) which gives the first approximation to v. Clearly the first approximation, though coarse, is not at all unreasonable.

The Wake of a Flat Plate

The wake of a flat plate provides a more severe test of the approximation. Fig. 3 shows the configuration; the liquid on the center line accelerates rapidly, requiring a large negative v-component to satisfy continuity. The inaccuracy in our estimate of v will accordingly be reflected fairly strong in the results.

The free-stream velocity U_∞ will be used as the reference velocity, U_0; the trailing edge boundary-layer thickness, δ, will be used as the reference length, L; and $\left(\dfrac{dP}{dx} - \mathrm{Fr}/\mathrm{Re}\right)$ will be set equal to zero. The system that we must solve is thus:

$$(u^2)_x = 2[u_y{}^2 + u_{yy}] \tag{10b}$$

with the following side conditions:

$$u_y(x, 0) = 0; \quad u(x, \infty) = 1.0$$
$$u(0, y) = \text{the Blasius profile} \tag{15}$$

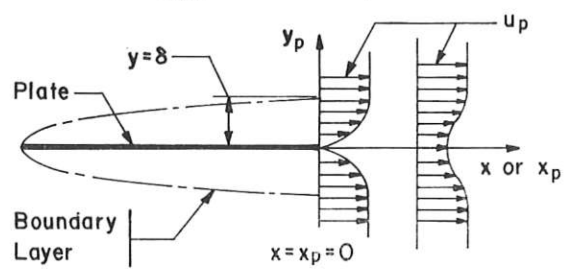
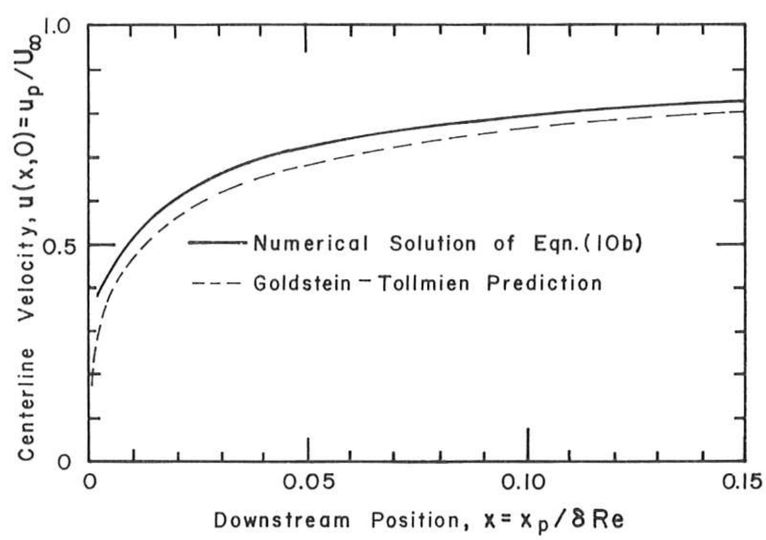


Fig. 3 The wake of a flat plate

Fig. 4 Velocities in the center of the wake of a flat plate



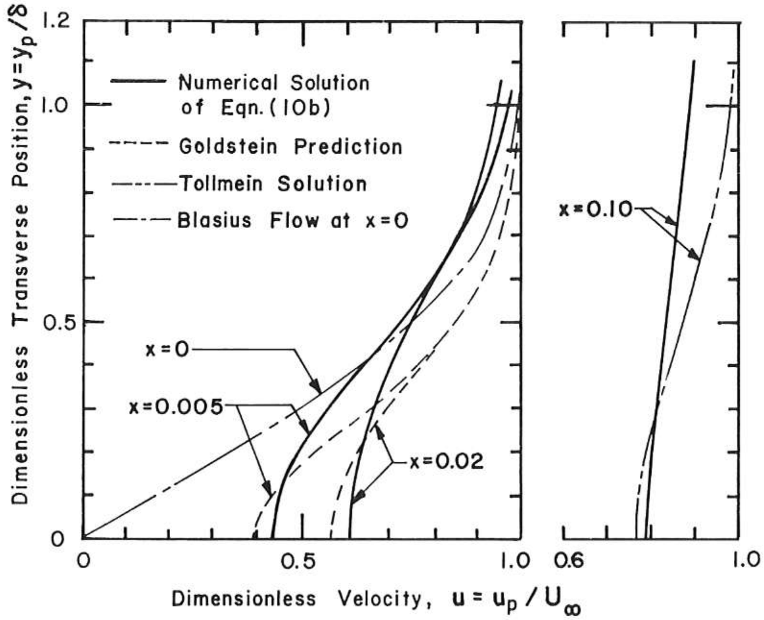


Fig. 5 Typical velocity distributions in the wake of a flat plate

This system was solved numerically on an IBM 360/50 computer. The Blasius profile was read in numerically, and a simple differencing scheme was employed to calculate the velocity profiles at subsequent downstream stations. Figs. 4 and 5 compare the present results with the early solution developed by Goldstein [5, 6] and Tollmien [7].

Fig. 4 shows that our center-line velocities are within 4 percent of these more accurate results, except for very small x's. Fig. 5 shows that, because v has been underestimated, our velocity profiles spread out somewhat more rapidly than with the more accurate integrations. On the other hand, the integral requirements (13) and (14) constantly corrected the exiting Poiseuille flow computation as it advanced in x.

Part II—Free Laminar Jet With Gravity and Surface Tension[2]

Analysis

Consider a free laminar jet leaving a Poiseuille tube oriented along the earth's gravity field. The tube may either discharge upward (Fr carries a minus sign in front of it) or downward (Fr carries a plus sign in front of it).

To work this problem we shall solve equation (11a) numerically subject to boundary conditions (12) and continuity condition (13). The momentum condition, equation (14), is no longer valid since gravity constantly increases momentum as the jet falls, and surface tension decreases momentum. Thus the momentum rate at any cross section can be equated to the exit momentum, less a surface tension term, and plus a gravity term.

$$\int_0^{\Delta_p} (2\pi r_p)(\rho u_p^2)dr_p = \frac{2}{3}(2\pi\rho R^2 \bar{U}_c^2) - (\text{surface tension term}) + \bar{u}_p \int_0^{x_p}\int_0^{\Delta_p} (2\pi r_p)(\rho g)dr_p \frac{dx_p}{u_p} \quad (16)$$

The surface tension term represents the contribution of compound curvature through pressure inside the jet. The net contribution will consist of two parts, one from circumferential tension and the other from axial tension

$$(\text{surface tension term}) = \pi\Delta_p^2\left[\frac{\sigma}{\Delta_p} - \frac{\sigma}{R_c}\right] \quad (17)$$

[2] This problem was treated in a different way by Duda and Vrentas [8] in a paper that appeared only very recently, and came to my attention after this paper underwent review. Duda and Vrentas developed a numerical solution based upon equations (4) and (5) and obtained strikingly similar results.

and the inverse axial radius of curvature, R_c^{-1}, (see Fig. 1) is given by

$$R_c^{-1} = \frac{d^2\Delta_p}{dx_p^2}\left[1 + \left(\frac{d\Delta_p}{dx_p}\right)^2\right]^{-3/2} \quad (18)$$

Substitution of equation (18) into equation (17) and the result into equation (16) yields, after nondimensionalization

$$\int_0^{\Delta} ru^2dr = \frac{2}{3} - \frac{\Delta^2}{\text{We}}\left[\frac{1}{\Delta} - \frac{\Delta_{xx}}{\text{Re}^2(1 + [\Delta_x/\text{Re}]^2)^{3/2}}\right] + \frac{\text{Re}}{\text{Fr}}\int_0^x \bar{u}\left[\int_0^{\Delta}\frac{r}{u}\,du\right]dx \quad (19)$$

This more complete statement of conservation of momentum must be used in place of condition (14).

Finally, we must evaluate dP/dx in the governing equation. The pressure gradient in the liquid resulting from surface tension will be obtained by differentiating and nondimensionalizing the bracketed term in equation (17). After substitution of equation (18) we obtain:

$$(u^2)_x = 2\left[u_r^2 + ([ru_r]_r)/r - \left\{\frac{2}{\text{We}}\frac{d}{dx}\left(\frac{1}{\Delta} - \frac{\Delta_{xx}}{\text{Re}^2(1 + [\Delta_x/\text{Re}]^2)^{3/2}}\right) - \frac{\text{Re}}{\text{Fr}}\right\}(1 + ru_r/2)\right] \quad (20)$$

Equation (20) was solved for u^2 on the computer using a simple

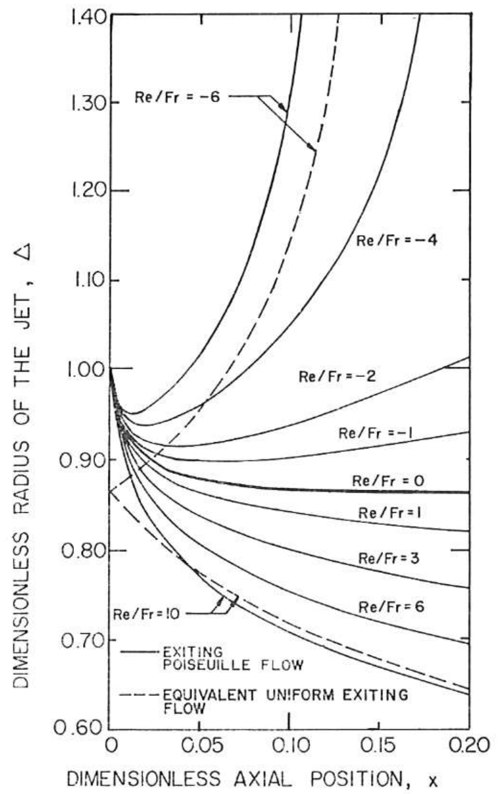


Fig. 6 Predicted jet radii for $|Re/Fr| \leq 10$ and negligible surface tension

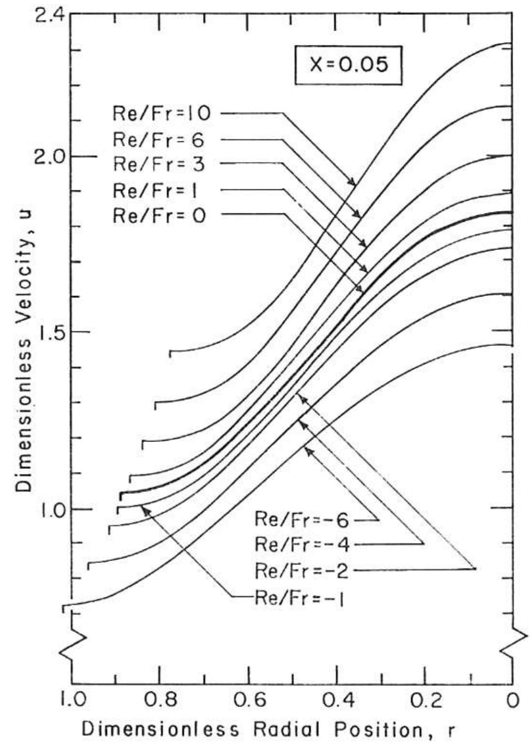


Fig. 7 Influence of gravity upon velocity profiles at a typical station ($x = 0.05$) with negligible surface tension

forward differencing scheme. After each step forward the dimensionless mass and momentum rates, $\int_0^\Delta rudr$ and $\int_0^\Delta ru^2dr$, were computed. The subsequent increments of radius and axial velocity were then stretched to make these rates satisfy conditions (13) and (19).

Calculated Results

The results of this computation depend upon three independent parameters, Re, Fr, and We. The Reynolds number is embedded in the dimensionless v and x variables, and it otherwise appears explicitly in the term Re/Fr and within the surface tension term. Reynolds number would also emerge as a third distinct parameter in the event that viscous dissipation played a role in the problem. Middleman and Gavis [9] showed that this effect would not be felt for $\text{Re} \geq 50$. We shall eliminate such low Reynolds numbers from consideration in the present study.

Middleman and Gavis also found that surface tension had no influence when $\text{We} \geq 100$. When this is the case Re/Fr completely controls the solution. The parameter Re/Fr has the significance of characterizing the ratio of gravity to viscous forces.

Fig. 6 shows a family of predicted jet profiles for $-6 \leq \text{Re/Fr} \leq 10$—that is, for both upward and downward directed jets. It is, of course, limited to $\text{We} \geq 100$ and $50 \leq \text{Re} \leq 1050$, where 1050 is the upper limit upon laminar Poiseuille flows. It is instructive to compare these curves with the elementary unidimensional theory:

The equivalent unidimensional flow would emerge from a $(\sqrt{3}/2)R$ radius jet at $\frac{4}{3}\ \bar{U}_e$. It would obey the following momentum and continuity expressions:

$$u_p{}^2 = 2gx_p + \frac{16}{9}\ \bar{U}_e{}^2; \quad \frac{u_p}{\bar{U}_e} = \left(\frac{R}{\Delta_p}\right)^2$$

When these are combined in dimensionless form we obtain

$$\Delta = \left(2\ \frac{\text{Re}}{\text{Fr}}\ x + \frac{16}{9}\right)^{-1/4} \tag{21}$$

Equation (21) is plotted for two cases in Fig. 6. It shows that the relaxation of the Poiseuille distribution initially governs the form of the profile. After that the curve takes the form predicted by equation (21), but slightly displaced in x.

A complete description of the velocity profiles over a range of (Re/Fr)'s would be quite cumbersome. By way of synopsis we shall only present selected computations. Fig. 7 shows the velocity profiles for several (Re/Fr)'s at a typical downstream station, $x = 0.05$. Fig. 8 shows how the center-line velocity varies with x for several (Re/Fr)'s. The relaxation of the center-line velocity into that of the equivalent unidimensional jet is illustrated in the figure for $(\text{Re/Fr}) = 10$. This is obtained from the momentum relation

$$u = \sqrt{2(\text{Re/Fr})x + 16/9}$$

Clearly the center-line velocity relaxes much more slowly than does the radius of jet.

Since surface tension adds two more parameters to the present problem—parameters of generally less importance to the early relaxation of the velocity profile—we shall only bring it into consideration in connection with specific experiments reported in the next section.

Experiment

The following elementary experiment was used to measure jet radii as a function of downstream position in an axial gravity field. Isopropanol was siphoned from a supply tank through 0.174-in-ID stainless steel tubing as shown in Fig. 9. The flow rate was measured and used to calculate Re and Fr. The physical properties of isopropanol are well documented as a function of temperature, which was measured during the runs. Jet diameters were scaled off sharp photographs made from a distance with a 35-mm camera and extension tubes. The straight downward run

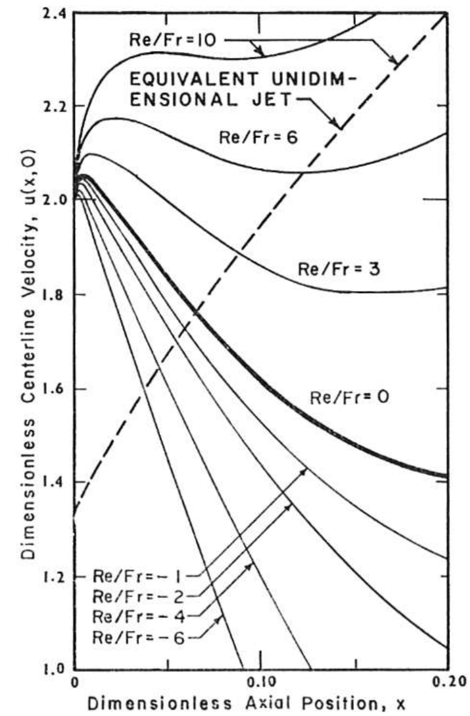


Fig. 8 Influence of gravity upon the center-line velocity of jets with negligible surface tension

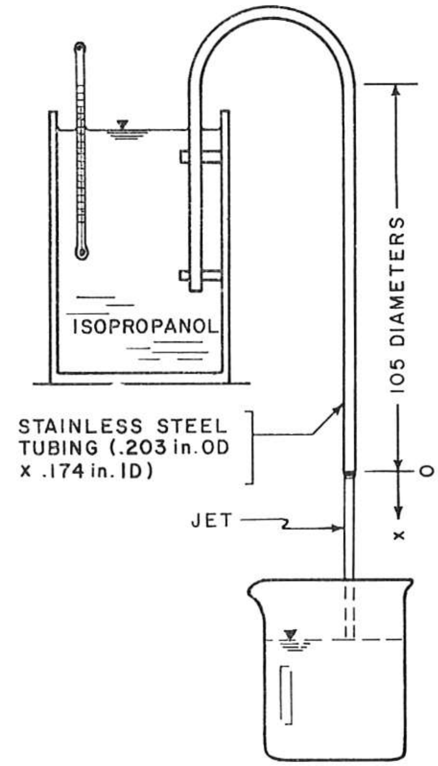


Fig. 9 Apparatus

of tubing was more than long enough to assure a fully developed exit velocity profile.

The results of the experiment were graphs of dimensionless jet radius against x. The error in measuring from enlargements of the photographs was about $\pm 1/2$ percent. Tube dimensions were measured within $\pm 1/4$ percent. Physical properties and $\bar{U}_e$ (which were used in computing Re, Fr, and We) were known within about 2 percent. The experiment was as simple as possible and generally free of error.

Fig. 10 shows a typical result. It includes the data photograph to scale in the x-coordinate, the measured profile in dimensionless coordinates, the numerical solution of equation (20), and the unidimensional approximation. The numerical solution agrees with the observed profile within the error of the experiment. The agreement is, in fact, so close that it is difficult to distinguish between the curves.

In this case, the isopropanol wetted the lip of the tube and there was no influence of contact angle of jet other than a great increase in Δ for $x \leq 0.001$. Fig. 11 shows one of the data photographs of Brun for water leaving a capillary tube. In this case the lip was not wetted. The resulting contact angle was such as to cause an immediate and severe contraction of the jet that cannot be predicted by equation (20). However, as $x \rightarrow 0.20$ (not shown in Fig. 11), both the data and the numerical solution converge on the same final Δ as obtained by Middleman and Gavis.

Numerical solutions for both Figs. 10 and 11 were also run without the surface tension term (i.e., for $(1/\text{We}) = 0$). The results showed little difference in the relaxation of the jet, but they converged to terminal values of Δ that were a little higher. Since Middleman and Gavis have adequately documented these terminal values, we shall not pursue surface tension effects further in this study.

Conclusions

1 An approximation to the transverse velocity component in the boundary-layer equations has been developed. It reduces

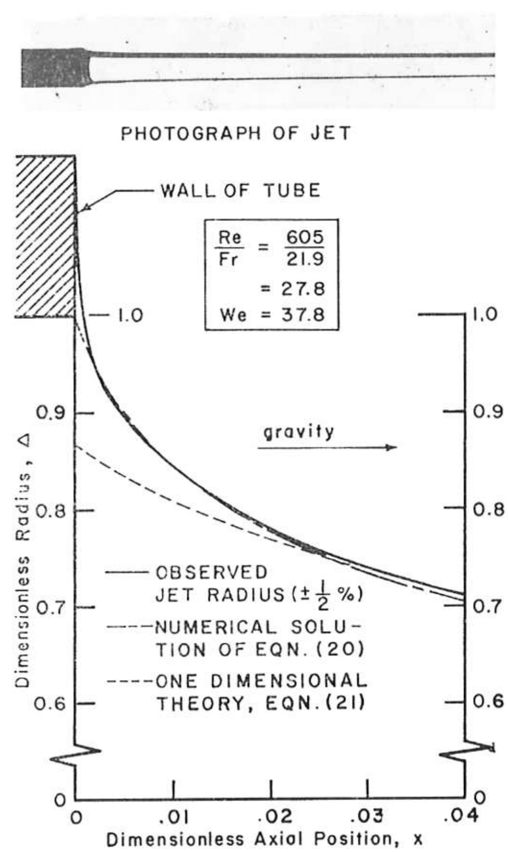


Fig. 10 Comparison of observed and predicted jet radii with gravity and surface tension

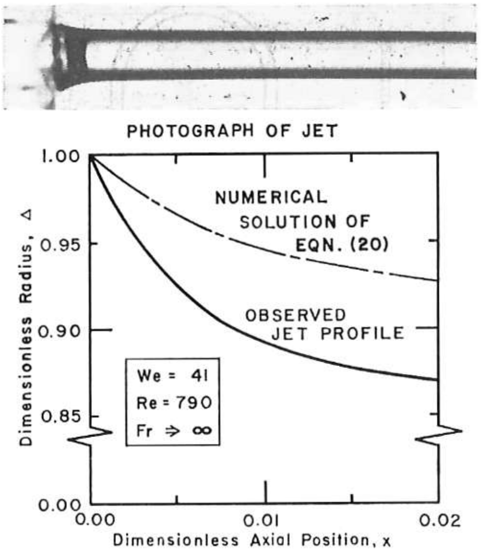


Fig. 11 Failure of numerical solution in a jet that is influenced by contact angle

the equations to a single numerically tractable equation, but it applies only in symmetrical flows.

2 The approximation is fairly good, and it can be made extremely good by adding integral conditions to satisfy conservation of mass and momentum.

3 The flow in exiting Poiseuille flows with an axial gravity force and with surface tension around the jet is described with great accuracy.

4 Liquid-solid contact angle at the discharge plane can completely invalidate the analytical solution if it occurs at low We.

Acknowledgments

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