An Extension of Statistical Mechanics to the Description of a Broad Class of Macroscopic Systems

By John H. Lienhard, and L. Berkley Davis, Jr., Dept. of Mechanical Engineering, University of Kentucky, Lexington, Kentucky

Introduction

The ideas to be developed here have their origins in two earlier papers [1] and [2]. In [1] the methods of Boltzmann statistics were used to predict the runoff of rain from a watershed. The second paper extended and generalized the prediction given in the first paper. It showed that the so-called *generalized gamma distribution*, which included most of the commonly used and physically plausible distribution functions as special cases, could be obtained from a simple statistical model. We shall first review these ideas briefly, and then formulate the 'thermodynamics' that would describe this family of non-molecular systems.

Suppose that a set of independent and distinguishable particles or events are distributed among values of some random variable, t. These particles (or events) obey the following physical limitations:

1. Their total number is fixed.

$$\sum_{i=0}^{\infty} \frac{N_i}{N} = 1 . \tag{1}$$

2. A βth moment, which we shall call $t_{rm\beta}$, is known. This can be a simple mean ($\beta = 1$), a root-mean-square, ($\beta = 2$, $t_{rm\beta} = t_{rms}$), or any other moment

$$\sum_{i=0}^{\infty} \frac{N_i}{N} t_i^{\beta} = (t_{rm\beta})^{\beta} . \tag{2}$$

3. The physical system to be described is such that there are g_i ways for the particle to be (or the event to occur) in the interval $[t_{i-1}, t_i)$. We shall restrict atten-

tion to systems for which g_i is of the form[1])

$$g_i = A\, t_i^{\alpha-1} = (A\, \Delta t^{\alpha-1})\, i^{\alpha-1} \tag{3}$$

where A is a constant and Δt is the increment of t_i.

What has been described is a system of distinguishable or 'localized' boltzons, with a fairly general sort of degeneracy and a more general specification of the variability of the distribution (equation (2)) than is used for Boltzmann statistics. The thermodynamic probability for such a system is well known, and in this case it takes the form:

$$W = N! \prod_{i=0}^{\infty} \frac{(A\, t_i^{\alpha-1})^{N_i}}{N_i!} \,. \tag{4}$$

Maximization of W subject to constraints (1) and (2) gives the expression for the distribution numbers — the (N_i/N)'s — that we can expect to obtain in the system. In the continuous limit this expression becomes the generalized gamma distribution,

$$f(t) = \left\{ \frac{\beta}{\Gamma(\alpha/\beta)} \left(\frac{\alpha}{\beta}\right)^{\alpha/\beta} \frac{1}{t_{rm\beta}} \right\} \left(\frac{t}{t_{rm\beta}}\right)^{\alpha-1} \exp\left[-\frac{\alpha}{\beta} \left(\frac{t}{t_{rm\beta}}\right)^{\beta} \right]. \tag{5}$$

It is important to recognize that the constants in equation (5) are not, in general, arbitrary. We are considering systems in which $t_{rm\beta}$ is known ahead of time — both its magnitude and the correct value of β. The constant, α, is the result of an *a priori* physical statement (possibly an assumption). By way of illustration, suppose that we wish to describe a t equal to the molecular speed, C. The average energy of a particle, $m\, C_{rms}^2/2$, is known, so condition (2) takes the form·

$$\sum_{i=0}^{\infty} \frac{N_i}{N} \left(\frac{m}{2}\, C_i^2\right) = \frac{m\, C_{rms}^2}{2} \tag{2a}$$

where m is the molecular mass. The degeneracy of C_i is known from quantum mechanical considerations to be

$$g_i = \frac{\pi}{2}\, i^2 \,. \tag{3a}$$

Thus $\beta = 2$ and $\alpha = 3$ for this case and the system described by equations (1), (2a) and (3a) is simply a Maxwellian ideal gas. Equation (5) reduces to

$$f(C) = \frac{1}{C_{rms}} \sqrt{\frac{54}{\pi}} \left(\frac{C}{C_{rms}}\right)^2 \exp\left[-\frac{3}{2} \left(\frac{C}{C_{rms}}\right)^2 \right], \tag{6}$$

the Maxwell molecular speed distribution.

A few of the other familiar distributions that are included as special cases of equation (5) can be listed briefly:
the *Wiebull distribution* $(\beta = \alpha)$

$$f(t) = \frac{\alpha}{t_{rm\alpha}} \left(\frac{t}{t_{rm\alpha}}\right)^{\alpha-1} \exp\left[-\left(\frac{t}{t_{rm\alpha}}\right)^{\alpha} \right], \tag{7}$$

---

[1]) This restriction will apply throughout this discussion. Possibly the whole endeavor can be opened to a broader physical model, but that is beyond our present scope.

the *rainfall hydrograph* [1] ($\beta = 2$)

$$f(t) = \frac{2}{\Gamma(\alpha/2)} \left(\frac{\alpha}{2}\right)^{\alpha/2} \frac{1}{t_{rms}} \left(\frac{t}{t_{rms}}\right)^{\alpha-1} \exp\left[-\frac{\alpha}{2}\left(\frac{t}{t_{rms}}\right)^2\right],$$ (8)

the *gamma distribution* ($\beta = 1$)

$$f(t) = \frac{\alpha^\alpha}{\Gamma(\alpha)} \frac{1}{t_{mean}} \left(\frac{t}{t_{mean}}\right)^{\alpha-1} \exp\left[-\alpha\left(\frac{t}{t_{mean}}\right)\right],$$ (9)

the *Rayleigh distribution* ($\beta = \alpha = 2$)

$$f(t) = \frac{2}{t_{rms}} \left(\frac{t}{t_{rms}}\right) \exp\left[-\left(\frac{t}{t_{rms}}\right)^2\right],$$ (10)

the *exponential distribution* ($\beta = \alpha = 1$)

$$f(t) = \frac{1}{t_{mean}} \exp\left[-\frac{t}{t_{mean}}\right],$$ (11)

and the *Maxwell molecular velocity distribution* ($\beta = 2, \alpha = 1$)

$$f(t) = \frac{1}{t_{rms}} \sqrt{\frac{2}{\pi}} \exp\left[-\frac{1}{2}\left(\frac{t}{t_{rms}}\right)^2\right].$$ (12)

Two Applications

Rainfall runoff. In Ref. [1], the 'event' under consideration was the elapse of time, t, between the falling of a raindrop on a watershed and its eventual arrival at a gaging station at the outlet of the watershed. A root-mean-square ($\beta = 2$) time of arrival was taken as the characteristic parameter of the watershed. The degeneracies were then established by the following physical considerations: In a long slender watershed the number of possible ways a raindrop can get to the gaging station increases in direct proportion to the distance it falls from the station. Thus $g_i \sim t_i$ and α is 2.

In a fan-shaped watershed, the area subtended by a given distance is proportional to the square of that distance. Thus the number of alternative paths for the raindrop is much larger and $g_i \sim t_i^2$. In this case α is 3. The resulting distribution of runoff was equation (8) with α equal to 2 or 3. The comparison of this prediction with measured runoff was quite good. Figure 1 shows typical comparisons from [1], for a reasonably fan-shaped watershed.

Income distribution. Another application currently under study is the use of the generalized gamma distribution to predict the distribution of personal annual income. The characteristic parameter of the distribution of income in any given year is the mean ($\beta = 1$) income for that year, $\bar{\$}$, where $t = \$$ will designate the annual income of an individual.

The degeneracy in this case takes on a somewhat subtle meaning. It represents the number of ways in which a man, who can earn $\$$ dollars per year, is able to earn

this income. Since income is probably closely correlated with a man's capability, we expect that more of the existing opportunities are within the grasp of the men who occupy the higher income levels. If the year is a 'fat' one, there will be many existing opportunities; if it is 'lean', there will be few. Accordingly we assume that

$$g_i \sim \$^{\alpha-1} \tag{3b}$$

where α is a kind of 'job accessibility number'. It varies from a small positive value in lean years to a high value in fat ones.

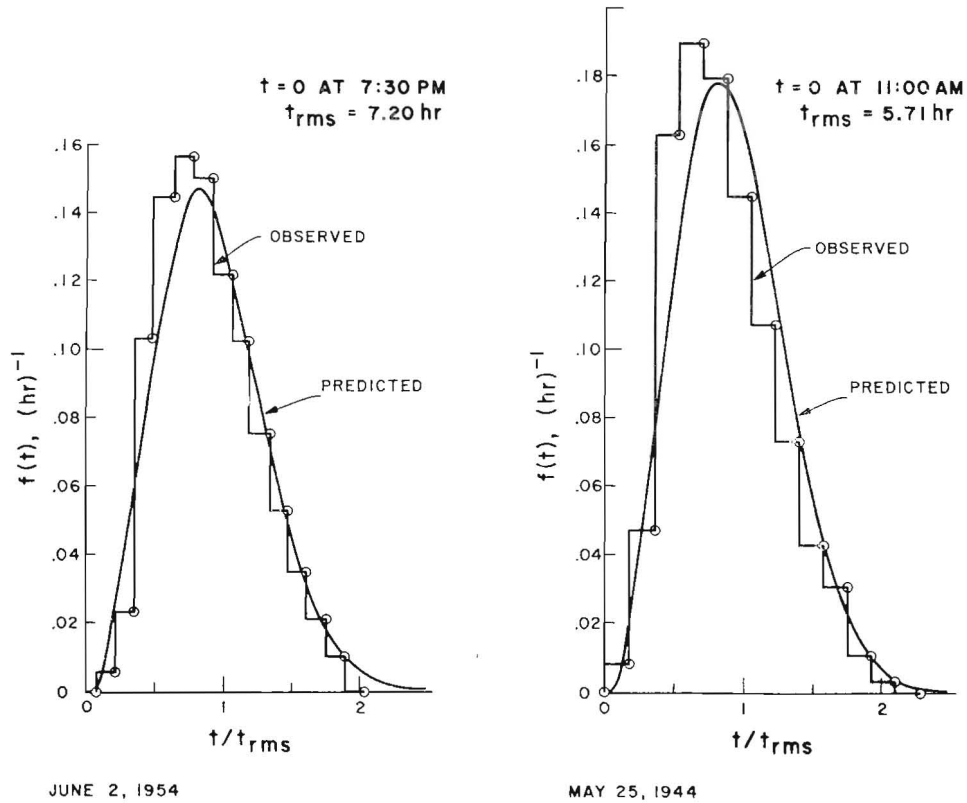JUNE 2, 1954   MAY 25, 1944

Figure 1
A typical comparison between predicted and observed runoff from a fan-shaped water shed in Mill Creek, Illinois (from reference [1]).

The result is the gamma distribution (equation (9)). Typical predictions are shown for the years 1935 and 1962 in Figure 2. At this writing, accurate descriptions of the words 'lean' and 'fat' are under consideration. However, there can be no doubting the relative wealth of these two years. The figure shows that for appropriate values of α (a large one for 1962 and a small one for 1935) the comparisons are quite good.

a) Distribution function for 1935-6 -- a "lean" year
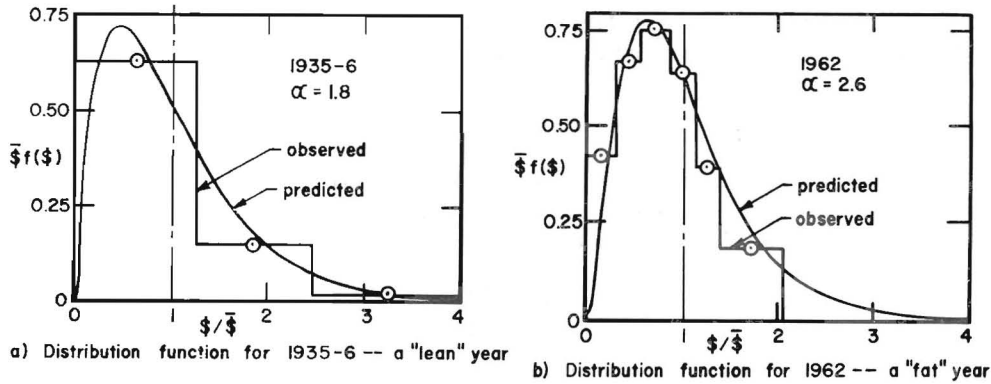
b) Distribution function for 1962 -- a "fat" year

Figure 2
Comparison of equation (9) with observed income data for a) a 'lean' year and b) a 'fat' year.

An interesting consequence of the theory is that for the limiting case of the wealthiest possible year (that is, the largest possible α) we can show

$$\underset{\alpha \to \infty}{\text{Limit}} \, f(\$) = \delta(1) \tag{13}$$

where $\delta(1)$ is the Dirac delta function with an argument equal to unity. Thus as times become very good, income tends to equalize at the mean value. In very bad times there are, on the other hand, far more people earning less than the mean and only a few very wealthy people who keep the mean up[2]).

In the preceding discussion we have found that independent distinguishable events or particles, subject to some simple constraints, obey the generalized gamma distribution. It turns out that if the particles are *in*distinguishable, and if $g_i \gg N_i$, then nothing will change in these results. The factor of $N!$ in equation (4) will be omitted, but equation (5) and everything that follows it will remain the same. This will be an important consideration in the subsequent development.

The Fundamental Equation of the System

We now wish to develop the *thermodynamic* description of systems which obey the generalized gamma distribution in a broader way than has been done before. Such a description must begin with the fundamental equations of such systems.

A fundamental equation for a macroscopic system is the expression from which all thermodynamic information may be obtained. It can take many forms; but for simple compressible systems in which the energy, U, the volume, V, and the number of particles (or moles), N, are independently specifiable it can be written as

$$\frac{S}{k} = \frac{S(U, V, N)}{k} \tag{14}$$

---

[2]) Thus Horatio Alger's story is not a parable of our own wealthy times – a fact that some conservatives often fail to recognize. It is of course important to recognize that equation (13) is an outgrowth of equation (3) which can be true only if free enterprise is a dominant force

where S is the entropy of the system and k is Boltzmann's constant. This expression in turn provides the link between the macroscopic and microscopic descriptions because

$$\frac{S}{k} = \ln W \ . \tag{15}$$

When the correct quantum mechanical and statistical description of a monatomic ideal gas is used for W in equation (15) the resulting fundamental equation is the important Sackur-Tetrode equation

$$\frac{S}{k} = N \left\{ \ln \left[\left(\frac{2\,m\,\pi\,k\,T}{h^2} \right)^{3/2} \frac{k\,T}{p} \right] + \frac{5}{2} \right\} \ . \tag{16}$$

where m is the mass of a molecule, h is Planck's constant, and T and p are the system temperature and pressure. Equation (16) is the form of the Sackur-Tetrode equation for a gas composed of indistinguishable monatomic particles, and it requires that $g_i \gg N_i$.

We can develop a fundamental equation equivalent to the Sackur-Tetrode equation for generalized gamma distribution systems. Such systems are analogous to monatomic ideal gases because only one mode of existence of t is acknowledged. For the analogy to hold, we must also view the events or particles as indistinguishable, but we have already noted that this will involve no loss of generality in our results. To obtain the fundamental equation, we divide the right-hand side of equation (4) by $N!$ and combine it with Stirling's approximation and equation (15). The result is

$$\frac{S}{k} = \sum_{i=0}^{\infty} N_i \ln \frac{g_i}{N_i} + N \ . \tag{17}$$

Using equation (5) in its finite form

$$\frac{N_i}{N} = \left\{ \frac{\beta}{\Gamma(\alpha/\beta)} \left(\frac{\alpha}{\beta} \right)^{\alpha/\beta} \frac{1}{t_{rm\beta}} \left(\frac{t_i}{t_{rm\beta}} \right)^{\alpha-1} \exp \left[-\frac{\alpha}{\beta} \left(\frac{t_i}{t_{rm\beta}} \right)^{\beta} \right] \right\} \Delta t \tag{18}$$

in equation (17) and making an appropriate transition from summation to integration, we obtain the analogue of the Sackur-Tetrode equation:

$$\frac{S}{k} = N \left\{ \ln \left[\frac{\Gamma(\alpha/\beta)\,A}{\beta(\alpha/\beta)^{\alpha/\beta}\,\Delta t\,N} \, t^{\alpha}_{rm\beta} \right] + \frac{\beta+\alpha}{\beta} \right\} \ . \tag{19}$$

Equation (19) is the fundamental equation of our system. The cell size, Δt, is left in this expression, just as the cell size $h^3/V = h^3\,p/NkT$ is left in equation (16). We shall have more to say about this point subsequently.

Equations of State

Equation (19) makes it possible to generate 'thermodynamic' properties for this class of new physical systems. This can be done by analogy with the classical thermodynamical property definitions

$$\frac{1}{k\,T} \equiv \left. \frac{\partial(S/k)}{\partial U} \right|_{V,N} ; \quad \frac{p}{k\,T} \equiv \left. \frac{\partial(S/k)}{\partial V} \right|_{U,N} ; \quad -\frac{\mu}{k\,T} \equiv \left. \frac{\partial(S/k)}{\partial N} \right|_{V,U} \tag{20}$$

where μ is the chemical potential. When the known function, S/k, is substituted in these definitions and the operations carried out the results are called *equations of state*.

In the new system, the meaning of N is unchanged. Since the energy of an ideal gas is really just a known moment of the distribution of particles, its proper analogue in the new system is $N\, t_{r\,m\,\beta}^{\mu}$. Thus we shall define

$$K = N\, t_{r\,m\,\beta}^{\mu}, \text{ analogous to } U \,. \tag{21}$$

Finally, an appropriate parameter equivalent to volume has to be identified. To do this let us first trace the role of volume in the ideal gas problem: equation (16) is derived by considering the distribution of momentum in a space which has the dimensions of (momentum)^3. But, by the dictates of quantum mechanics, phase space with the units of (action)^3 is broken into cells of uniform size, h^3. Volume is accordingly a *comparative factor* between the *absolute* cell size in phase space and the variable cell size in momentum space. Thus,

$$\Delta t = (2\,\Delta\,[m\,C])^3 = \frac{h^3}{V} \tag{22}$$

where m is the particle mass and C is its speed.

Now by way of fixing attention for a moment on one of the new systems in particular, let us return to the income distribution problem. In the same way that molecules have access to an increasing number of modes of energy storage as the number of dimensions of physical space increases, so too do men have access to increasing modes of income as the job accessibility number, α, is increased. Pursuing the notion that degeneracy is proportional to the $(n-1)^{\text{th}}$ power of the random variable in an n-dimensional physical space, we can view α as equal to the number of dimensions of the 'physical' space of the income problem.

We might then envision a kind of economic phase space – \$ in one direction, and some kind of a metric for extension in accessibility, in the α other directions. The independently specifiable volume in this phase space will be called H^α. Although H^α is like h^3 in its function, no science of quantum mechanics exists to specify its value in this case. Therefore H will remain an arbitrary constant. By analogy with equation (22) we then define a 'volume', B, for income, or for any other of the new systems, as

$$B = \frac{H^\alpha}{\Delta t} \tag{23}$$

so

$$\Delta t = \frac{H^\alpha}{B}. \tag{24}$$

The equations of state that we wish to propose are then:

$$\frac{1}{k\,\mathfrak{J}} \equiv \left.\frac{\partial(S/k)}{\partial K}\right|_{B,N}; \quad \frac{\mathfrak{p}}{k\,\mathfrak{J}} = \left.\frac{\partial(S/k)}{\partial B}\right|_{K,N}; \quad \frac{\mathfrak{m}}{k\,\mathfrak{J}} = -\left.\frac{\partial(S/k)}{\partial N}\right|_{K,B} \tag{25}$$

where the script variables $\mathfrak{J}$, $\mathfrak{p}$ and $\mathfrak{m}$ are properties that will be analogues to T, p and μ, for the new system. If we first substitute equation (24) into equation (19) to

get the appropriate form of the fundamental equation

$$\frac{S}{k} = N \left\{ \ln \left[\frac{\Gamma(\alpha/\beta) \, A}{\beta(\alpha/\beta)^{\alpha/\beta} \, H^\alpha} \, \frac{K^{\alpha \beta} \, B}{N^{1+\alpha/\beta}} \right] + \frac{\alpha + \beta}{\beta} \right\} \quad (19a)$$

then we can immediately evaluate the equations of state for the new systems. They are

$$\frac{1}{k \, \mathcal{J}} = \frac{\alpha}{\beta} \, \frac{N}{K}, \quad (26)$$

$$\frac{\mathcal{P}}{k \, \mathcal{J}} = \frac{N}{B}, \quad (27)$$

and

$$-\frac{m}{k \, \mathcal{J}} = \frac{S/k}{N} - \frac{\alpha + \beta}{\beta}. \quad (28)$$

It is a straightforward exercise to show that for the Maxwell-Boltzmann case, equation (19) becomes the Sackur-Tetrode equation and equations (26), (27) and (28) reduce to the familiar equations of state : $k \, T = m \, C_{rms}^2/3$, $p/k \, T = N/V$ and $\mu_i'/k \, T = 5/2 - S/k \, N$. This exercise gives us confidence to proceed in the application of our new fundamental equation. We wish first to look at the thermodynamic laws for our new systems. Then we shall attempt to put the whole notion into somewhat nicer perspective with the help of an example.

The Laws of Thermodynamics

Since the laws of thermodynamics have their origins in the statistical behavior of molecules, and since the properties we have described evolve from the same statistical description, the same laws that apply to molecules will apply to the present systems. Accordingly we shall paraphrase the laws of thermodynamics to apply here.

The phenomenological laws of classical thermodynamics are not well suited to this use because they are couched in terms of such processes as heating and working. It seems an unnecessary exercise to develop far-fetched analogies for these processes when the laws can be recast in such a way as to describe only equilibrium end points. Accordingly we shall adopt the axioms of Callen [3] and restrict them to single component systems. The laws that govern the properties, S, B, K and N, are

1. There exist particular states (called equilibrium states) of simple systems that, macroscopically, are completely characterized by B, K and N.

2. There exists a function, S, of the extensive parameters, B, K and N, defined for all equilibrium states and having the following property: The values assumed by the extensive properties in the absence of an internal constraint are those that maximize S over the manifold of constrained equilibrium states.

3. S for a composite system is additive over the subsystems S is continuous, differentiable, and a monotonically increasing function of K.

The fourth postulate (or third law of thermodynamics) is not relevant to the present discussion since it requires that the system pass through certain condensation(s) to arrive in a zero-entropy state that is not described by equation (19).

The first two postulates permit us to paraphrase the conventional relation,

$$dU = \sum_{i=0}^{\infty} N_i \, d\varepsilon_i + \sum_{i=0}^{\infty} \varepsilon_i \, dN_i = -p \, dV + T \, dS \,, \tag{29}$$

with

$$dK = \sum_{i=0}^{\infty} N_i \, dt_i^\beta + \sum_{i=0}^{\infty} t_i^\beta \, dN_i = -\mathcal{p} \, dB + (k \, \mathcal{J}) \, d \, \frac{S}{k} \,. \tag{30}$$

The second postulate also says that

$$d \left(\frac{S}{k} \right)_{isolated} \geqslant 0 \tag{31}$$

and equation (19) clearly satisfies all the requirements of the third postulate

Example – Rainfall Runoff

We wish now to talk about the thermodynamics of a watershed. To do so we shall take the view that if we change the conditions within one watershed, or go from one watershed to another, with a given number, N, of raindrops, then what we are doing is equivalent to changing the thermal environment of a group of N molecules.

Let us consider only fan-shaped watersheds for the moment. Then $\alpha = 3$ and $\beta = 2$, and equation (26) gives the 'temper' [3], $1/k \, \mathcal{J}$, as

$$\frac{1}{k \, \mathcal{J}} = \frac{3}{2 \, t_{rms}^2} \,. \tag{32}$$

Consider then two ways to change the characteristic time of discharge, t_{rms}, of the watershed:

One way would be simply to scale the watershed up in size. This would stretch the t_i's without changing the distribution numbers and would be analogous to *work*. This could be done either by making everything physically larger or by other means – reducing the force of gravity for example

A second way to change t_{rms} would be by changing the distribution numbers without altering the scale. This might be accomplished physically by planting trees or by defoliating the land. It would occur normally with the changing seasons. Such changes would increase or decrease W or S/k, and would be analogous to *heat*.

Either kind of change would generally result in a change of temper as equation (32) indicates. If we are to make the first kind of change (the result of 'work') meaningful we must say what B is in physical terms. We shall simply take it to be the area [4], Λ, and allow H to accommodate itself to this definition. The 'pressure' is then

$$\mathcal{p} = \frac{k \, \mathcal{J} \, N}{\Lambda} = \frac{2 \, t_{rms}^2}{3} \, \frac{N}{\Lambda} \tag{33}$$

---

[3] The word 'temper' is used here in accordance with a growing feeling that it is an appropriate name for the Lagrangrian multiplier, $1/k \, T$, in conventional statistical thermodynamics.

[4] Actually it might be feasible to do something more elaborate with the scaling of the watershed. The gravity, for example, might be included for its influence on the effective size.

in accordance with equations (27) and (32). Then equation (30) gives

$$d(N\, t_{rms}^2) = -k\, \boldsymbol{J}\, N\, d\ln\!\Lambda \cdot\cdot k\, \boldsymbol{J}\, d\, \frac{S}{k} \tag{34}$$

If we include equation (32) and integrate the result we obtain

$$\frac{\Lambda\, t_{rms}^3}{(\Lambda\, t_{rms}^3)_0} = \exp\frac{S - S_0}{k\, N} \tag{35}$$

where S_0 is the entropy for which $\Lambda\, t_{rms}^3 = (\Lambda\, t_{rms}^3)_0$.

Equation (34) shows that changes of time of runoff are the sum of a pure scale effect and a pure disordering effect. The latter is the only one we would consider in any single watershed; however, the former would hold considerable interest if we set out to compare similar watersheds of different sizes. Equation (35) is an equation of state that relates area, rms time, and entropy to one another. We must be careful to recognize that equation (34) is restricted to reversible processes, or processes in which equilibrium is maintained at each point, and the fundamental equation (19a) is always valid. Equation (35) is not restricted in this way, however, because it describes only equilibrium end points.

Equation (35) shows the relation between area and t_{rms} for watersheds that can be transformed into one another by reversible processes. This, as we have noted, would entail simple changes in the scale of watersheds. In such cases,

$$t_{rms} \sim \Lambda^{-1/3}\,. \tag{36}$$

Reference [1] gives data for two hydrographs for each of two Illinois watersheds in late Spring storms – one for each watershed in the early 1940's and one each in the early 1950's. If we call the watersheds, no. 1 and no. 2, then $(\Lambda_1/\Lambda_2)^{1/3} = 1.17$. In the early 1940's, (t_{rms_1}/t_{rms_2}) was 1.11 which implies that the watersheds were comparable in surface. But in the early 1950's (t_{rms_1}/t_{rms_2}) was 1.4 owing to an increase in t_{rms_1}. This implies that some kind of gross change in the surface of no. 1 had occurred.

While this illustration may be somewhat thin in terms of data, it serves its purpose which is to show how a thermodynamic analysis of a stochastic system might generate new information or new modes of analysis. Finally, let us see how it might be possible to make direct application of equation (31).

Suppose that N raindrops were constrained to run off of only one side of a fan shaped reservoir. Then suppose we 'released this constraint' and made the other side available to these drops as well (i.e., suppose the fan were to be opened). The function $K = N\, t_{rms}^2$ would be constant and equation (34) would give

$$\left.\frac{\partial\,(S/k)}{\partial\Lambda}\right|_{K,\,N} = \frac{N}{\Lambda} = \text{positive} \tag{27a}$$

indicating that S/k would, indeed, increase.

Conclusion

The statistical mechanical description used to represent the behavior of ideal gases can be extended to describe the behavior of systems composed of macroscopic elements. These elements are treated as indistinguishable, degenerate boltzons.

Such a description has been carried out subject to conditions given in equations (1), (2) and (3). A fundamental equation for this broad class of systems has been derived and the equations of state have been written. Two examples have been described to show how these systems of macroscopic elements might exhibit behavior analogous to thermodynamic behavior of gases.

It will be in the exploration of examples that the importance of this idea can finally be properly assessed. We have touched upon two physical problems in this paper, and are turning our attention to several others. Consideration is presently being given to a more complete development of the income problem. Further thought has and is also being given to such diverse matters as: 1. the grade-point average distribution of a system of students, 2. the size distribution of spray droplets [4], 3. the failure of components in either time or stress and 4. the distribution of numbers of airplanes shot down by WWI air aces.

The use of statistical mechanics to describe molecules does, after all, require the abstraction of thermodynamic meanings to a point that is very general indeed – to a point at which the description is no longer closely wed to the nature of molecules – to a point at which something altogether different from molecules might just as well be under description. It is our hope that the broader use of statistical mechanics to predict the distribution of all sorts of random variables will fruitfully be undertaken in the future.

Nomenclature

A	undetermined constant of proportionality
B	generalized volume defined as $H^\alpha/\Delta t$
C, C_i, C_{rms}	molecular speed, subscripts i and rms denote i^{th}-level, and root-mean-square, values
$f(t)$	distribution function for t
g_i	degeneracy described in context of equation (3)
H	generalized Planck's constant
h	Planck's constant
i	summation index $i = 1, 2, 3, \ldots$
K	generalized energy defined as $N\, t_{rm\beta}^\beta$
k	Boltzmann's constant
m	molecular mass
$\mathcal{m}$	generalized chemical potential, see equation (25)
N, N_i	number of particles, events, or moles. Subscript i denotes i^{th} level
p	thermodynamic pressure
$\mathcal{p}$	generalized pressure, see equation (25)
T	thermodynamic temperature
$\mathcal{J}$	generalized temperature, see equation (25)
t	random variable
t_i	value of t at the ith level
$t_{rm\beta}, t_{rms}, t_{rm\alpha}, t_{mean}$	moments of t defined by equation (2). Subscripts indicate the βth, root-mean square, α^{th} and mean values
S, S_0	entropy. Subscript, 0, indicates a reference value

U	thermodynamic energy
V	thermodynamic volume
W	thermodynamic probability
α	number of dimensions in generalized space
β	number which determines the moment in the generalized gamma distribution
ε_i	energy of the ith state
A	area of a watershed
μ	thermodynamic chemical potential
$\$,\ \$_i$	annual income. Subscript denotes i^{th} level.

References

[1] J. H. Lienhard, *A Statistical Mechanical Prediction of the Dimensionless Unit Hydrograph*, J. Geophys. Res. *69*, no. 24, 5231 (1964).

[2] J. H. Lienhard and P. L. Meyer, *A Physical Basis for the Generalized Gamma Distribution*, Quart, appl. Math. *25*, no. 3, 330 (1967).

[3] H. B. Callen, *Thermodynamics* (John Wiley and Sons, New York 1960).

[4] J. H. Lienhard and J. B. Day, *The Breakup of Superheated Liquid Jets*, J. Basic Engr, *92*, no. 3, 515 (1970).

Zusammenfassung

Die statistische «thermodynamische» Beschreibung einer Gruppe von neuen Systemen wird entwickelt. Drei Bedingungen beschreiben die Gruppe: 1. Jedes System hat eine bekannte Bevölkerung. 2. Die Elemente des Systems haben eine einzige wichtige willkürlich verteilte Eigenschaft, und eines der Momente dieser Verteilung ist bekannt. 3. Die Entartung der verteilten Eigenschaft ist proportional zu einer einfachen Potenz dieser Eigenschaft. Das Maxwell-Boltzmannsche ideale Gas ist ein Sonderfall dieser Gruppe.

Die thermodynamische Beschreibung der Gruppe der Systeme umfasst die Verteilungsfunktion, eine fundamentale Gleichung und Zustandsgleichungen. Zwei typische Anwendungen (eine des Regenflusses, und eine der Verteilung des Einkommens) werden beschrieben.